

**THE MODIFIED COVERING PROBLEM
ON PATHS AND TREES**

by

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TABLE OF CONTENTS

	Page
LIST OF ILLUSTRATIONS.	6
LIST OF TABLES	7
ABSTRACT	8
1. INTRODUCTION	9
1.1 Definitions	9
1.2 Motivations for the Modified Covering Problem	10
2. LITERATURE REVIEW.	13
2.1 Motivations for the Conditional Covering Problem	13
2.2 Formulation.	15
2.3 Algorithm and Heuristic Development.	17
2.4 Conclusions.	19
3. THE MODIFIED COVERING PROBLEM (MCP)	21
3.1 Classification within the Schilling Taxonomy	21
3.2 Mathematical Formulation	22
3.3 Feasible Solution to the Modified Covering Problem	24
3.4 Reformulation with Assumptions	25
3.5 Reformulation Advantages	26
4. OPTIMALITY AND COMPLEXITY	28
4.1 Preliminary Definition	28
4.2 Bounding the Optimal Solution	29
4.3 Solution Complexity	31
5. THE MCP ON PATHS WITH UNIFORM LINK DISTANCES	33
5.1 Optimal Solution Patterns	34
5.2 Analytical Solution and Formal Algorithmic Statement	39
5.3 Complexity Analysis for Uniform Link Distances.	41

TABLE OF CONTENTS - *Continued*

	Page
6. THE MCP ON PATHS WITH NON-UNIFORM LINK DISTANCES	42
6.1 Optimal Solution Patterns	42
6.2 Simple Path Algorithm	49
6.3 On the Optimality of All Independent Pairs	51
6.4 Obtaining a Lower Bound for the Algorithm.	52
6.5 Revised Path Algorithm	56
6.6 Revised Path Algorithm Complexity.	57
7. THE MCP APPLIED TO TREES	65
7.1 Preliminary Definitions.	65
7.2 Tree Decomposition and Motivation for Bounding	66
7.3 Intersecting Paths at Interior Nodes	67
7.4 Intersecting Paths at Interior/Exterior Nodes.	71
7.5 Determining the Bounding Effects of Each Node	76
7.6 Closed Form Bound for the MCP for Trees	77
8. CONCLUSIONS AND FUTURE RESEARCH	78
8.1 Conclusions.	78
8.2 Future Research	79
WORKS CITED	80
ADDITIONAL REFERENCES	82

LIST OF ILLUSTRATIONS

	Page
FIGURE 4.1 – Smallest Network with Feasible MCP Cover	28
FIGURE 4.2 – Largest Network with Smallest Feasible MCP Cover	30
FIGURE 5.1 – Independent Pair of Facilities on a Path	35
FIGURE 5.2 – Independent Trio of Facilities on a Path	36
FIGURE 6.1 – Independent Group of more than Three Facilities	44
FIGURE 6.2 – Decomposing an Independent Group of more than Three Facilities	45
FIGURE 6.3 – Two Consecutive Independent Trios.	46
FIGURE 6.4 – Three Consecutive Independent Pairs	47
FIGURE 6.5 – Another Three Consecutive Independent Pairs	47
FIGURE 6.6 – More Than One Independent Trio in Optimal Solution	48
FIGURE 6.7 – Decomposition Attempt Resulting in Non-optimal Solution	49
FIGURE 6.8 – Upper Bound with Independent Pairs	52
FIGURE 6.9 – Improvement to Solution over All Independent Pairs	52
FIGURE 6.10 – Three Independent Pairs	54
FIGURE 6.11 – Improved Recomposition	55
FIGURE 6.12 – Plot of Complexity Curve and Approximations	61
FIGURE 6.13 – Logarithmic Plot of Complexity Curve and Approximations . . .	62
FIGURE 6.14 – Plot of Successful Curve Fit	63
FIGURE 6.15 – Logarithmic Plot of Successful Curve Fit	64
FIGURE 7.1 – No Improvement with Intersecting Paths.	68
FIGURE 7.2 – One Facility Improvement with Intersecting Paths.	69
FIGURE 7.3 – Two Facility Improvement with Intersecting Paths.	69
FIGURE 7.4 – No Improvement with Intersecting Paths.	72
FIGURE 7.5 – One Facility Improvement with Intersecting Paths.	73
FIGURE 7.6 – Two Facility Improvement with Intersecting Paths.	74

LIST OF TABLES

	Page
TABLE 5.1 – Covering Abilities of Independent Groups of Facilities	37
TABLE 5.2 – Decomposition of Independent Groups of Facilities	37
TABLE 6.1 – Combinations of z_{UB}	50
TABLE 7.1 – Maximum Facility Redundancy at a Node where Paths Intersect . .	76

ABSTRACT

The Modified Covering Problem (MCP) is introduced and theory is developed for solving it on paths and trees. First, the Modified Covering Problem is defined as a subset of the Conditional Covering Problem, and motivations are proposed for its study. Next, a literature review examines relevant, published material.

The MCP is then formulated as a binary integer program, followed by an examination of the characteristics of its feasible solutions, optimality, and overall complexity. A polynomial algorithm is developed for the solving the MCP on paths with uniform link distances, and solving within 20% of optimality on paths with non-uniform link distances. Next, an exponential algorithm is developed to solve non-uniform link distance problems to optimality. The theory is then further expanded to construct an algorithm to develop strong upper and lower bounds for the optimal solution on trees with non-uniform link distances.

CHAPTER 1

INTRODUCTION

This chapter introduces the modified covering problem and motivations for its study.

1.1 Definitions.

The *set-covering problem* (SCP) seeks to minimize the number of facilities while locating them in order to cover all demands.

The *conditional covering problem* (CCP) has the same objective with an additional set of constraints: each facility that is established must be covered by another established facility. This inter-facility relationship requirement was first defined in 1984 by Moon and Chaudhry [1984] when they introduced the conditional covering problem.

Finally, the proposed *modified covering problem* (MCP) is a special case of the conditional covering problem. The modified covering problem has three specified characteristics for which it differs from the conditional covering problem. First, the facilities have symmetric capabilities. The covering radius is constant for all possible facility locations. Second, the MCP is only formulated as a cardinality version, where there is no difference in the cost of opening facilities on different sites. Although the CCP may have identical facility costs, cardinality is not a specified construct of the

model. Finally, the set of possible facility locations is the same as the set of demand locations. These three characteristics are applicable to the motivations for the Modified Covering Problem.

1.2 Motivations for the Modified Covering Problem.

Two categories of applied problems motivate the study of the modified covering problem. The first category of problems occurs when a facility cannot serve a demand at its own location. This occurs when a facility has both a minimum and a maximum covering radius.

One military example within this category is the positioning of artillery batteries in a peace enforcement area of operations. Artillery pieces have a lower and upper bound on their effective range; when placed on a map, their maximum (360°) coverage area looks roughly like a donut with a very small hole in its center. Due to the lower bound, artillery units can provide indirect fire for other units, but they cannot provide indirect fire for their own defense. As a result, artillery units must receive indirect fire support from other units.

This artillery example fits the model for the modified covering problem but not the conditional covering problem. First, it seeks to locate artillery units with identical characteristics. They have the same set-up and operations cost as well as range. The

artillery units will only be located within the confines of the base camps (demands) because the artillery units need the force protection they provide. Finally, the model may be modified to incorporate capacities in order to reduce the complexity of target management.

The second category of motivation exists when the occurrence of a unit demand would either render useless or destroy a collocated facility. This catastrophic type of demand occurs in military or national security environments.

As an example within this second category, recent U.S. Congressional legislation [1996] funded the training of Weapons of Mass Destruction (WMD) Civil Response Teams. These medical teams are located in 120 major cities across the United States in order to rapidly respond to biological or chemical terrorist incidents. Although the team locations were most likely determined through a combination of logical prioritization (largest cities) and politics, this problem is an ideal application for the modified covering problem. WMD Civil Response Teams must be located in order to cover high probability terrorist targets, but teams cannot necessarily cover the city in which they are located. A terrorist biological or chemical attack on a city may render its own team incapable of performing its mission. Therefore, a different team must cover any city in which a team is located.

This example also fits the model for the modified covering problem but not the conditional covering problem. First, it seeks to locate teams with identical characteristics. They would have the same operational and training cost. As these teams are formed from personnel with other, primary occupations as medical and emergency personnel, cost-of-living differences between regions can be ignored; those costs are born by the primary employer. Assuming similar transportation assets, the teams also have identical covering radii. In order to provide access to transportation hubs, the teams will be located in major cities. Accordingly, the set of feasible facility locations is the set of demands (cities). Finally, the model may be modified to incorporate capacities in order to balance the planning load among WMD teams.

CHAPTER 2

LITERATURE REVIEW

All of the previously published literature focuses on the conditional covering problem, not the modified covering problem. However, their similarities make the literature worthy of review because it can provide intuition for solution approaches to the modified covering problem. As a subset of the conditional covering problem, the modified covering problem can be solved with any techniques developed for the CCP, though the converse is not true.

This chapter explores motivations for the conditional covering problem, shows its mathematical formulation, and discusses previous efforts at algorithmic and heuristic development.

2.1 Motivations for the Conditional Covering Problem.

Many of the proposed motivations for the study of the conditional covering problem are of questionable validity. Moon and Chaudhry [1984] first referred to the problem as “a double set-covering problem,” referring to the dual set of requirements to (1) cover all demands and (2) cover all facilities. However, the problem and its potential motivations have often been confused with the multiple-covering problem formulation proposed by Van Slyke [1982], in which each demand must be covered by at least two facilities.

One of Moon and Chaudhry's [1984] flawed motivations for the CCP is the "establishing backup facilities for fire fighting or ambulance services" to serve demands when another facility is busy. As proposed, one valid formulation of this application is a multiple-covering problem that requires each demand to be within the service capability of at least a primary facility and a back-up facility. Another valid formulation is to establish a set of primary (demand covering) facilities and another set of backup (primary facility covering) facilities. Neither formulation is a valid motivation for the CCP. The first formulation does not actually require inter-facility distance requirements. The second formulation requires back-up facilities to have to demand responsibilities in order to move into a station whose service providers are occupied. In the CCP, demand coverage and facility coverage are not mutually exclusive responsibilities.

Another flawed motivation in previous literature is the opening of two new commercial facilities (e.g. stores) to serve a given market with the option of closing one of them later [Chaudhry, 1993]. This is also a multiple-covering problem because -- although it initially divides the market between the facilities -- both must be capable of covering all demands in order to allow for a closure. The inter-facility distance is unimportant to this application, so long as all demands are covered by both facilities. The corresponding solution to a CCP formulation would not guarantee that all demands could be covered when the second facility is closed. Once again, in the CCP, facilities do not have an exclusive role of either covering demands or other facilities; they may perform any combination of both roles.

Moon and Chaudhry [1984] proposed only one valid motivation for the conditional covering problem: inter-facility support. They outlined the practical application of requiring transshipment capability between distributors or warehouses that have non-overlapping service regions. This could be useful in order to offset shortages in one region with excess inventory in a nearby region within a specified amount of time.

Successive journal articles and conference proceedings to Moon and Chaudhry's [1984] initial work merely repeat the same set of motivations, both valid and invalid.

2.2 Formulation.

The following notation is necessary to formulate the mathematical model for the conditional covering problem (CCP):

Decision Variables:

$x_j =$ 1 if a facility is located at site j , 0 otherwise;

Sets and Set Notation:

$N = \{j, k: j, k=1, 2, \dots, n\}$, the index set of available facility sites, $x_j=x_k$;

$M = \{i: i=1, 2, \dots, m\}$, the index set of demands;

Note: In earlier publications for the original conditional covering problem [Chaudhry Moon, McCormick, 1987], the index sets for facility sites and customers were reversed. This change in formulation can be confusing to researchers, but this thesis will maintain the more common, and recent, notation as outlined above.

Other Variables:

- $c_j =$ fixed cost of opening a facility at site $j \in N$
 $a_{ij} =$ 1 if a facility at site j can cover demand i , 0 otherwise
 $b_{kj} =$ 1 if a facility at site j can cover a facility at site k , 0 otherwise
 $(b_{kk} = 0 \text{ for all } k \in N)$

Formulation:

$$\text{minimize} \quad z = \sum_j c_j x_j \quad (2.1)$$

$$\text{subject to} \quad \sum_j a_{ij} x_j \geq 1 \quad \forall i \in M \quad (2.2)$$

$$\sum_j b_{kj} x_j \geq x_k \quad \forall k \in N \quad (2.3)$$

$$x_j \in \{0,1\} \quad \forall j \in N \quad (2.4)$$

The objective function (2.1) represents the total cost of the facilities opened. When the cardinality version exists, where $c_j = 1$ for all $j \in N$, the objective function represents the total number of facilities opened. Meanwhile, if the cost of each possible facility is equal but not one ($c_j = d \neq 1$ for all $j \in N$, where d is a constant), the solution that minimizes the cost of open facilities will also minimize the number of facilities, and the costs may be ignored.

The first set of constraints (2.2) requires that each demand be covered by at least one facility. The second set of constraints (2.3) requires that each located facility be covered by at least one other facility. Finally, the third set of constraints (2.4) restricts the decision variables to zero-one integer variables.

2.3 Algorithm and Heuristic Development.

Most efforts to solve the conditional covering problem have been mere extensions of set covering problem techniques. Additionally, almost all of the previous literature focuses on algorithm and heuristic development for solving the cardinality (unweighted) version of the conditional covering problem. Unless otherwise specified, the following techniques were examined only for the cardinality CCP.

2.3.1 Algorithms.

When the conditional covering problem was first proposed [1984], Moon and Chaudhry attempted to solve it with a combination of linear relaxation of constraint set (2.4) to the following form:

$$0 \leq x_j \leq 1 \quad \forall j \in N \quad (2.5)$$

Then they introduced a single cutting plane constraint in an attempt to resolve any non-integer solution values. This was a direct extension of the technique employed by Toregas, et. al. [1971] to effectively solve the set covering problem. However, the technique was not as reliable or effective for solving the conditional covering problem.

In 1995, Moon and Lotfi published their efforts to improve the implicit enumeration algorithm for solving the conditional covering problem. They extended the work of Balas and Ho [1980] on the set covering problem: using Lagrangian-based lower bounds and heuristic-determined upper bounds in order to more efficiently fathom non-optimal solutions.

2.3.2 Heuristics.

Chaudhry, Moon, and McCormick [1987] proposed a set of seven greedy heuristics to solve the conditional covering problem, tested them on a set of 259 sample problems, and compared the results. Each heuristic was a modification of those for set covering problems as proposed and examined by Chvatal [1979], Johnson [1974], and Lovasz [1975].

Then, in 1990 Moon examined the weighted conditional covering problem and attempted to develop bounds for a simple greedy heuristic. This heuristic was an extension of the weighted set covering heuristic examined by Chvatal [1979]. Moon

showed that the heuristic result could not be bounded by anything other than an absolute worst-case ($z \leq \sum_j c_j$) value.

Next, Chaudhry [1993] proposed and tested two additional heuristics for the conditional CCP. These were developed out of intuition from the previous heuristics from 1987, but they did not improve upon their performance.

Finally, Lotfi and Moon [1994, 1997] improved upon the simple heuristics by incorporating facility exchange procedures into their techniques. This work was an extension of the parallel efforts by Vasco and Wilson [1986] on the set-covering problem. As intuition would suggest, their heuristics for the conditional covering problem were an improvement over the simpler greedy heuristics without exchange.

2.4 Conclusions.

Some lessons are worth extraction from studying the efforts of previous authors to examine the conditional covering problem. Throughout the literature, the authors modified techniques for the set-covering problem in order to approach the conditional covering problem. While the lack of expending independent, creative effort at developing new algorithms and heuristics may appear as an advantage, it may also result in overlooking simpler, more effective techniques. Additionally, although some of the

modified techniques were useful (heuristics), others failed to work when applied to the CCP (LP relaxation).

In approaching the modified covering problem (MCP), this thesis will focus on independently developing solution approaches rather than modify existing SCP techniques and ‘hope’ for success.

CHAPTER 3

THE MODIFIED COVERING PROBLEM

This chapter provides classification and mathematical formulation for the general (network) model of the modified covering problem (MCP). It then defines requirements for feasibility, and simplifies the model formulation with the set of assumptions required for a feasible solution.

3.1 Classification within the Schilling Taxonomy.

In 1993, Schilling, Jayaraman, and Barkhi defined a useful taxonomy for covering problems in facility location to help categorize future research. Accordingly, the modified covering problem (MCP) is classified as follows.

Topology. The MCP utilizes a network for spatial representation and deterministic values for a distance/time metric between nodes. However, the application for which the problem is applied can be planar with Euclidean, rectilinear, or other metric. The only requirement is that the inter-site distances can be used to deterministically answer the question: “Can a facility at site n cover a demand (or facility) at site s ?” in order to transform the problem into a network.

Demands. Demands are discrete, deterministic, and static. The MCP seeks to locate facilities in order to have the capability to cover a unit demand anywhere; it does not consider multiple demands that would incorporate travel times and service times of facility assets in order to form queues.

Facilities. Available facility locations are finite and reliable, with infinite capacities and uniform costs among available facility locations. This equates to symmetry among the network. If a facility at site r can cover a facility at site s , then a facility at site s can cover a facility at site r . Also, only one facility may be located at an available site.

3.2 Mathematical Formulation.

The modified covering problem is initially formulated with notation similar to the CCP. Deviations from the CCP formulation are shown here in boldface type:

Decision Variables:

$x_j =$ 1 if a facility is located at site j , 0 otherwise;

Sets and Set Notation:

$N =$ $\{j, k: j, k=1, 2, \dots, n\}$, the index set of available facility sites, $x_j=x_k$ **$N=M$** ;

$M =$ $\{i: i=1, 2, \dots, m\}$, the index set of demands;

Other Variables:

$a_{ij} =$ 1 if a facility at site j can cover demand i , 0 otherwise

$b_{kj} =$ 1 if a facility at site j can cover a facility at site k , 0 otherwise

($b_{kk} = 0$ for all $k \in N$)

Formulation: Expressed mathematically, the modified covering problem can be formulated similar to the conditional covering problem [Lotfi and Moon, 1994].

$$\text{minimize} \quad z = \sum_j x_j \quad (3.1)$$

$$\text{subject to} \quad \sum_j a_{ij} x_j \geq 1 \quad \forall i \in M \quad (3.2)$$

$$\sum_j b_{kj} x_j \geq x_k \quad \forall k \in N \quad (3.3)$$

$$x_j \in \{0,1\} \quad \forall j \in N \quad (3.4)$$

The objective function (3.1) represents the total number of facilities opened. Similar to the conditional covering problem, the first set of constraints (3.2) requires that each demand be covered by at least one facility; the second set of constraints (3.3) requires that each located facility be covered by at least one other facility; and the third set of constraints (3.4) restricts the decision variables to zero-one integer variables.

3.3 Feasible Solutions to the Modified Covering Problem.

Within the general framework of the problem, it is important to define the characteristics of feasible solutions. In order for a feasible solution to exist, the following properties must hold true:

- 1) Every demand must be within the covering radius of at least one possible facility location *that meets criteria #2*.
- 2) Every facility location *that is selected to cover demands* must be within the covering radius of at least one other possible facility location.

This inter-relationship between the criteria reflects the initial difficulty in determining if a feasible solution exists by inspection alone. Since determining the desired facility locations for criteria #2 requires determining a desired solution, an easier set of screening criteria is useful.

In order to reduce the problem complexity and inspect for feasibility, perform the following simple operations. First, eliminate any possible facility location that cannot be covered by at least one other possible facility location. Any such facility could not be part of a feasible solution. Then apply the following simpler set of criteria:

- 1) *Every demand must be within the covering radius of at least one possible facility location.*
- 2) *Every possible facility location must be within the covering radius of at least one other possible facility location.*

Although this set of criteria seems stricter, it does not modify the problem at all. It simply reduces the number of infeasible solutions.

3.4 Reformulation with Assumptions.

In applied engineering, not every problem has a feasible solution. For this reason the requirements for a feasible solution are not always listed as assumptions during initial formulation. However, their incorporation as assumptions can simplify the mathematical formulation. With the following set of assumptions:

$$\sum_j a_{ij} \geq 1 \quad \forall i \in M \quad (3.5)$$

$$\sum_j b_{kj} \geq 1 \quad \forall k \in N, (b_{kk}=0) \quad (3.6)$$

$$N \subseteq M \quad (3.7)$$

A feasible set is guaranteed to exist and the mathematical formulation can be rewritten. Taking advantage of the similarities between the [A] matrix and [B] matrix, and the fact that the set of facility sites is a subset of the set of demands, renumber the set of M

demands such that the first n demand sites correspond to the first n facility sites. After this is complete, the problem can be reformulated as:

Modified	<i>minimize</i> $z = \sum_j c_j x_j$	(3.8)
Covering	<i>subject to</i> $\sum_j b_{ij} x_j \geq 1 \quad \forall i \in N, (b_{jj}=0)$	(3.9)
Problem	$x_j \in \{0,1\} \quad \forall j \in N$	(3.10)

This formulation has the same objective function (3.8). Meanwhile, the first set of constraints (3.9) ensures demands are met by a facility that is not co-located. The final constraint set (3.10) still restricts the decision variables to zero-one integer variables.

3.5 Reformulation Advantages.

This reformulation (1) reduces the problem to two sets of indices, i and j ; reduces the number of constraints by n ; and allows the reduction of constraint sets (3.9) to a set that looks like the following when shown in matrix form:

$$\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ \dots \\ 1 \end{pmatrix} \quad (3.11)$$

In turn, the manipulation of this matrix and set of constraints is much easier than working with the original formulation. For a general network, the modified covering problem can take advantage of the (1) Row Feasibility, (2) Row Dominance, and (3) Column Dominance Rules [Francis, McGinnis, White, 1992] used in set-covering problems to reduce the number of demands and facility sites that must be considered in order to determine the optimal solution.

It remains important to develop better bounds on the optimal solution in order to improve the efficiency of enumerative algorithms like branch and bound or implicit enumeration.

CHAPTER 4

OPTIMALITY AND COMPLEXITY

This chapter derives simple bounds for optimal solutions, and examines the worst-case complexity of the problem for general networks as motivations for developing algorithms for paths and trees.

4.1 Preliminary Definition.

It is important to define a network for which a minimum size feasible set exists as a component of bounding the optimal solution. Such a network is shown in Figure 4.1. For this network $|N|=2$, $|M|=2$, and the arc length between the nodes is less than the covering radius of a facility ($r>1$). A diagram is shown below, where “ \bigcirc ” represents a demand and “ \times ” represents the location of a facility.

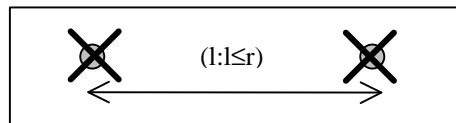


Figure 4.1 – Smallest Network with Feasible MCP Cover

This network with $z=2$ could have a large number of demands, so long as they are within r distance of either of the two facilities shown in Figure 4.1. It is also noteworthy that a larger MCP with a feasible solution may simply consist of several small, disjointed networks like the one in Figure 4.1.

4.2 Bounding the Optimal Solution.

The optimal solution for the modified covering problem is the feasible set with the minimum number of facilities. For the general case of a network with a feasible solution, the optimal solution has a lower bound of ($Z_{LB}=2$) facilities and an upper bound of ($Z=N$) facilities. The lower bound occurs on the minimum size network for which a feasible solution exists, as shown in Figure 4.1. The upper bound of ($Z_{UB}=N=2G$) occurs when the problem consists entirely of $|G|$ sub-networks ($g=1, 2, 3 \dots G$) where $N_g=2$, $M_g=2$, the sub-network has a feasible solution C_g , and $N=\sum_g(N_g)$. Of note, it follows that $M=\sum_g(M_g)$. Therefore, by simple inspection to ensure the validity of the assumption that a feasible solution exists, the optimal solution can be bounded by:

$$2 \leq Z^*_{MCP} \leq N \quad (4.1)$$

Since the MCP is a subset of the cardinality SCP (with $N=M$), we can derive better upper and lower bounds from the SCP's optimal solution. Given a problem with a feasible solution for the modified covering problem, it also has a feasible solution for the set-covering problem. Of note, visual inspection for a feasible SCP solution – checking to confirm that every demand can be covered by at least one possible facility location – results in the following bounds on the SCP optimal solution. (Equation 4.2)

$$1 \leq Z^*_{SCP} \leq N \quad (4.2)$$

Examining the most favorable case, given a feasible solution to the SCP, it is possible that all facilities are covered by another facility. An example network for which this holds true is also shown in Figure 4.2.

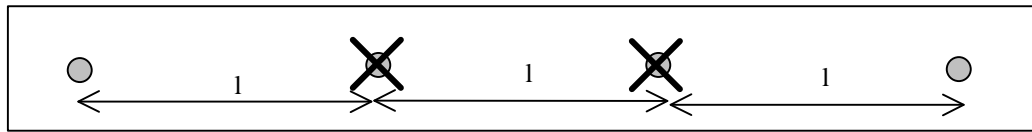


Figure 4.2 – Largest Network with Smallest Feasible MCP Cover

In this example, for any covering distance of $(2l > r \geq l)$, two facilities are required. The solution shown in Figure 4.2 (alternative SCP optima exist) is also a solution to the corresponding MCP. Therefore, the cardinality MCP has a lower bound of $Z^*_{MCP} \geq Z^*_{SCP}$.

The cardinality SCP also provides an upper bound for the cardinality MCP. For Figure 4.2, for any covering distance of $(r \geq 2l)$, the SCP requires only one facility. Meanwhile, the cardinality MCP requires two facilities in order to provide inter-facility covering. This simple worst-case provides an upper bound of $Z^*_{MCP} \leq 2 Z^*_{SCP}$.

Therefore, by finding the optimal cardinality SCP solution to a problem for which a feasible CCP solution exists, the MCP optimal solution can be bounded by:

$$Z^*_{SCP} \leq Z^*_{MCP} \leq 2 Z^*_{SCP} \quad (4.3)$$

When combining the bounds from visual inspection and the application of SCP procedures, we can further bound the optimal solution to the MCP by:

$$\text{MAX } \{2, Z^*_{\text{SCP}}\} \leq Z^*_{\text{MCP}} \leq \text{MIN } \{N, 2 Z^*_{\text{SCP}}\} \quad (4.4)$$

4.3 Solution Complexity.

A review of algorithm complexity is necessary before undertaking development efforts. Of note, the following definitions apply, where $O(\bullet)$ represents the worst-case number of operations that must be performed on a problem of size n in order to complete the algorithm that determines the optimal solution:

<u>Algorithm Time</u>	<u>Example of $O(\bullet)$</u>
Polynomial	$O(an^b)$ where a, b are constants
Exponential	$O(c2^n)$ where c is a constant

For a large problem – and corresponding large value for n -- it is desirable to have a polynomial time algorithm.

It is worthwhile to examine the costly order of magnitude for one of the simplest algorithms: enumeration. For the MCP – a zero-one integer program with n possible

facilities and m demands – there are 2^n possible solutions for which constraints must be checked for feasibility (n calculations), an objective value determined (1 calculation), and the objective value compared to the best previously determined solution (1 calculation for $2^n - 1$ solutions). Although the operations are simple, the total number of operations is a non-polynomial $(2^n(n+2)-1)$. As the exponential term, 2^n dominates when $n \rightarrow \infty$, the enumeration algorithm has an exponential complexity of $O(2^n)$.

Due to the high complexity of such algorithms on general networks, it is desirable to first examine the modified covering problem on paths and trees. The development of polynomial algorithms for optimizing the MCP on paths and trees may result in the following advantages:

- 1) If successful, these technique(s) can then be applied to the minimum-spanning tree of a network in an attempt to improve the bound on its optimal solution.
- 2) The technique(s) may provide improved, formal intuition for developing solution heuristics to solve the MCP on networks.

CHAPTER 5

THE MODIFIED COVERING PROBLEM ON PATHS

WITH UNIFORM LINK DISTANCES

In the field of operations research, among others, one paradigm has repeatedly proven successful when developing theory. First, examine and solve simple instances of problems. Then, build on the simple theory to examine and develop theory to solve problems that are more complex.

This chapter examines a first set of instances on paths: when all links are of uniform distance, l . Nodes represent the set of demands and the set of feasible facility locations. Meanwhile, available facilities continue to have uniform covering radius, r . For these instances, it is a requirement that $(l \leq r)$; otherwise, a feasible solution does not exist because inter-facility covering cannot be accomplished. The integer ratio, R , is the ratio between covering radius and link distance, rounded down to the nearest integer. This integer ratio is denoted by: $R = \lfloor r/l \rfloor$. Additionally, the optimal number of facilities is represented by z .

5.1 Optimal Solution Patterns.

Theorem 5.1. For the Modified Covering Problem on a path with uniform link distances, the optimal solution will consist entirely of independent pairs of facilities and at most one independent trio of facilities.

Definition: An *independent pair* is a set of two facilities that cover each other, but neither one covers – or is covered – by any facility outside the pair. Similarly, an *independent trio* is a set of three interdependent facilities that cover each other but are independent from all other facilities.

As defined by the term ‘independent,’ each facility is covered by at least one other within the group. To clarify the covering relationships, consider an independent trio of facilities: A, B, and C, located in order along a path. Facilities A and B cover each other; facilities B and C cover each other; but facilities A and C do not cover each other. This trio of facilities is ‘independent’ because they do not rely on facilities external to the group for their own coverage.

The proof for Theorem 5.1 focuses on a dual problem to the MCP: maximize the number of demands covered along a path by a given number of facilities. When examining the dual, it is shown that for a given number of facilities, they can cover the

longest uniform-link distance path when grouped as all independent pairs and at most one independent trio.

Proof. The proof consists of showing three components.

(1) There are no facilities placed in groups smaller than two (2). A ‘group’ of only one independent facility violates feasibility requirements for the solution: each facility must be covered by at least one other facility.

(2) There are no facilities placed in groups larger than three (3). This component is shown by examining the capabilities of independent sets of “k” facilities.

a) Two Independent Facilities. For a problem with ratio R , a set of two independent facilities can cover at most $c=(3R+1)$ demands. A diagram is shown below, where the **X** represents the location of a facility.

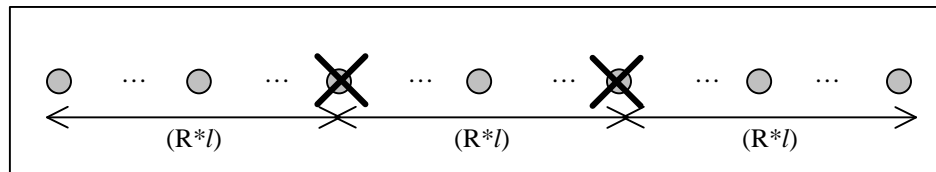


Figure 5.1 – Independent Pair of Facilities on a Path

Note that since $r \geq (R \cdot I)$, the facilities support each other and the demands at up to R links away from either facility are covered. This is the optimal

layout for two independent facilities. If the two facilities were located any closer to each other, fewer facilities would be covered. If the facilities were located any farther from each other, they would not be able to satisfy the inter-facility covering requirement.

- b) Three Independent Facilities. For a problem with ratio R , a set of three independent facilities can cover at most $(c+R)$ demands, where $c=(3R+1)$.

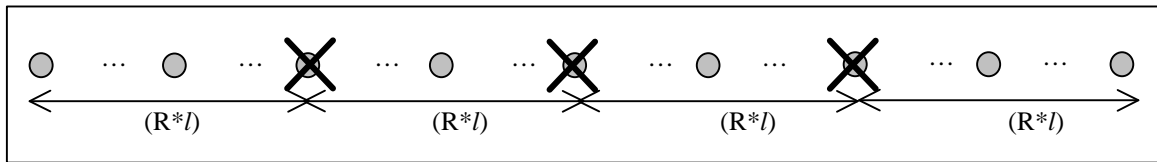


Figure 5.2 – Independent Trio of Facilities on a Path

This is the optimal layout for three independent facilities. Similar to the previous example, a smaller inter-facility would decrease the number of covered demands, and a greater inter-facility distance would violate the inter-facility covering requirement.

- c) Four or More Independent Facilities. For a problem with ratio R , if a set of more than three independent facilities is constructed by locating an additional facility at the last covered node, they can provide coverage as shown in Table 5.1:

Facilities	Length of Path Covered	Number of Demands Covered
2	$3R/$	$3R+1$
3	$4R/$	$4R+1$
4	$5R/$	$5R+1$
...		
N	$(N+1)R/$	$(N+1)R+1$

Table 5.1 – Covering Abilities of Independent Groups of Facilities

Given the dual goal of maximizing the total number of demands covered by a fixed number of facilities, no sets of four or more facilities will be established as independent groups. Instead, they will be decomposed into groups of two and three in order to cover more demands with the same number of facilities. Some simple examples of group decomposition are shown in Table 5.2.

Facilities	Original No. of Demands Covered	➔	Independent Pairs	Independent Trios	Number of Demands Covered
4	$5R+1$	➔	2	0	$6R+2$
5	$6R+1$	➔	2	1	$7R+2$
6	$7R+1$	➔	3	0	$9R+3$

Table 5.2 – Decomposition of Independent Groups of Facilities

If $R < 1$, there is no feasible solution. A facility would have to be located on every demand in order to cover them, but the inter-facility covering could not be achieved. Since $R \geq 1$ for any instance with a feasible solution, the optimal solution will be decomposed into sets of independent pairs and independent trios of facilities.

- (3) There is no more than one group of facilities of size three (3) in an optimal solution. By contradiction, assume that an optimal solution has two independent trios. These trios of facilities each cover $(4R+1)$ demands for a total of $(8R+2)$ demands. By decomposing them into three independent pairs, they can each cover $(3R+1)$ demands for a total improvement to $(9R+3)$ demands. Because a solution with more than one independent trio can be decomposed in order to cover more demands, it cannot be optimal.

Although it is easy to visualize the recomposition of two adjacent independent trios of a solution into three adjacent independent pairs, such a recomposition can occur even if the independent trios are not adjacent. Because the path has uniform link distances, independent groups can be rearranged in any order and cover the same length of the path. Accordingly, two ‘candidate’ groups for recomposition can be rearranged as adjacent groups, then recomposed into independent pairs and at most one independent trio. This operation can be performed recursively until the optimal solution pattern is obtained.

The results of the proof have additional implications. Specifically, optimizing the dual – maximize the total number of demands covered by a fixed number of facilities – is equivalent to (a) maximize the number of independent groups of facilities for a fixed number of facilities. It is also equivalent to (b) maximizing the number of uncovered links for a fixed number of facilities... the links in between independent groups of facilities. Finally, it is equivalent to (c) minimizing the number of facilities to cover a fixed number of nodes.

5.2 Analytical Solution and Formal Algorithmic Statement.

Given a path with uniform distance, l ; uniform facility covering radius, r ; and a finite number of demands, $n=|N|$; the number of facilities required can be determined by the following analytical equations:

$$R = \lfloor r/l \rfloor \quad (5.1)$$

$$c = (3R+1) \quad (5.2)$$

$$z^* = z(r, l, n) = \begin{cases} 2 & 2 \leq n \leq c \\ 3 & c < n \leq c+R \\ 4 & c+R < n \leq 2c \\ 5 & 2c < n \leq 2c+R \\ 6 & 2c+R < n \leq 3c \\ 7 & 3c < n \leq 3c+R \\ 8 & 3c+R < n \leq 4c \\ \dots & \dots \end{cases} \quad (5.3)$$

Expressed in algorithmic form, the optimal number of facilities can be determined by applying the following steps:

Uniform Link Distance Algorithm

- 1) Calculate integers R , c , and n such that:

$$R = \lfloor r/l \rfloor \qquad c = (3R+1) \qquad n = |N|$$

- 2) Calculate integers a and b such that:

$$b = \lceil (n/c) \rceil \qquad a = (b-1)$$

- 3) Apply the following rules to determine the optimal number of facilities:

IF ($a=0$)

THEN $z^*=2$

ELSE IF ($n > ac + R$)

THEN $z^*=2b$

ELSE $z^*=(2b-1)$

- 4) In order to locate the facilities, start at one end of the path and place an independent pair of facilities to cover each consecutive set of c demands. If placing an odd number of facilities, the last independent group located will be an independent trio covering the last (up to) $c+R$ demands.

Note that the “at most one” independent trio can be located anywhere along the path. This algorithm places it at the end of the path only though its recursive technique of placing independent pairs until a decision must be made whether to

- (a) place a final independent pair or (b) change the last independent pair to an independent trio in order to cover the path.

5.3 Complexity Analysis for Uniform Link Distances.

The complexity for solving the Modified Covering Problem on a path with uniform link distances is determined by making the five calculations (R, c, n, b, a) and at most 2 comparisons to determine the optimal solution pattern. Then the algorithm starts at one end of the path and considers each node only once for whether or not to locate a facility in accordance with the optimal pattern. This is a polynomial complexity of $O(\bullet)=O(n+7)$ or $O(\bullet)=O(n)$.

CHAPTER 6

THE MODIFIED COVERING PROBLEM ON PATHS WITH NON-UNIFORM LINK DISTANCES

This chapter expands the effort to examine paths for instances when the link lengths l_{ij} may vary from link-to-link, but always satisfies the condition ($l_{ij} \leq r$). Otherwise, the distance between that set of nodes i and j is insurmountable, and the problem is decomposed into two smaller problems. When ($l_{ij} > r$), a forest of paths is established, and each path may be solved as an independent location problem. Of note, the number of paths (separable location problems) in the forest is equal to $(1 + |(\text{links with } l_{ij} > r)|)$. Of course, each path in the forest must have at least two (2) nodes in order for a feasible MCP solution to exist.

Consider a path that has non-uniform link distances, contains at least two (2) nodes, and satisfies the condition ($l_{ij} \leq r$) for all links on the path.

6.1 Optimal Solution Patterns.

Theorem 6.1. For the Modified Covering Problem on a path with non-uniform link distances, an optimal solution (not necessarily a unique optimum) will consist entirely of independent pairs of facilities and independent trios of facilities. An optimal solution

may have more than one independent trio, but will not require consecutive independent trios of facilities along the path.

Using the same definitions for independent pairs and independent trios, the proof for Theorem 6.1 also focuses on a dual problem to the MCP: maximize the number of demands covered along a path by a given number of facilities. In proving the solution patterns to this dual problem, we can apply them to minimize the number of facilities for a given path.

Proof. This proof consists of showing three components:

(1) There are no facilities placed in groups smaller than two (2). Once again, the location of one independent facility violates the inter-facility covering requirement.

(2) There are no facilities placed in groups larger than three (3). Any time a independent group of $n=|N|$ facilities exists, it should be decomposed into a combination of:

$$n=2a+3b \tag{6.1}$$

where a represents an integer number of independent pairs and b represents an integer number of independent trios. This decomposition can only increase the number of demands covered by a given number of facilities; it cannot decrease it.

Consider the example shown in Figure 6.1. This figure depicts an independent group with at least four facilities on Points A, B, C, and E. The facility group size can be even larger with additional points to the right or left. However, we are only concerned with the effect of decomposing it by imposing a ‘break.’ Also, additional demands are located to the left of Point A and to the right of Point F along the path.

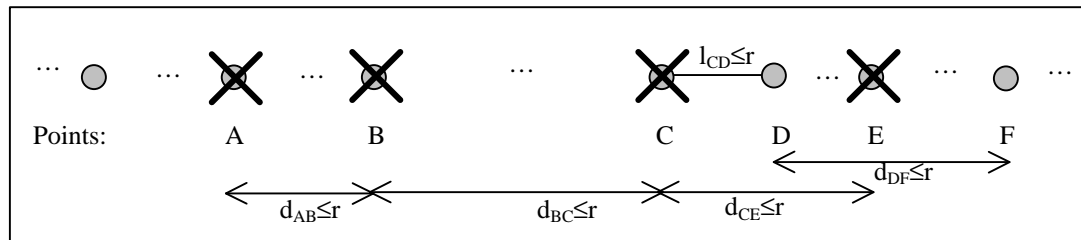


Figure 6.1 – Independent Group of more than Three Facilities

Note that each demand is covered and each facility is covered by at least one other facility in the group.

Now consider the effects of decomposition. Since Point C is covered by the facility at Point B, the facility originally at Point C can be moved to Point F, where Point F is the farthest point away from – but within covering radius

distance of – Point D, the first point not covered by the facility at Point B.

(Shown in Figure 6.2.)

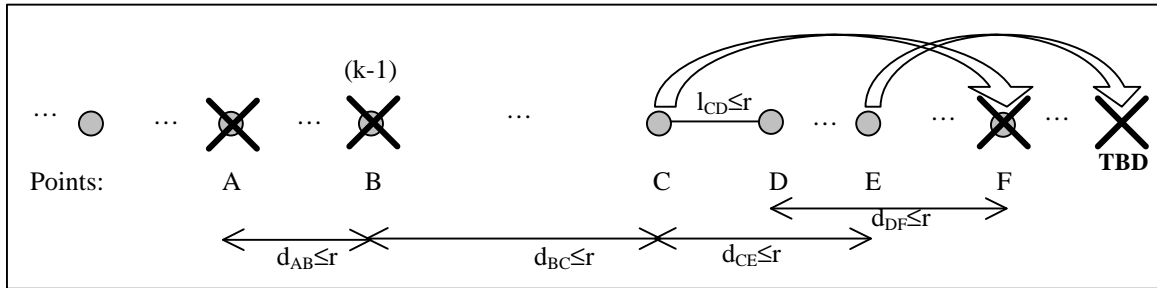


Figure 6.2 – Decomposing an Independent Group of more than Three Facilities

By locating the facility at Point F, the set of points up through Point F will cover at least one (1) more facility (G) and at least a distance greater by d_{CF} than if the facility had been located at Point C. The interspersion of demands to the right of Point F will determine where the next facility (formerly at Point E) may be located. Two extreme cases may exist:

- 1) Best Case. The decomposition is productive overall, resulting in more covered demands. The upper bound on improvement is not determined by a closed form solution, but on the inter-nodal distances to the right of Point F.
- 2) Worst Case. The move does not result in more covered demands. One such example is if Point G is the final point on the path and $d_{EG} \leq r$. In

this case, decomposition did not result in an improvement in the number of demands covered. (Point G was covered prior to decomposition.)

The argument is logically extended to an independent group of five (decomposed into a pair and trio), six (decomposed into three pairs or two trios), and so forth. Therefore decomposing any independent groups of size four or larger into a combination of independent pairs and independent trios can increase the number of demands covered, but cannot decrease the number of demands covered.

(3) An optimal solution will not require consecutive independent trios of facilities.

There is, however, no explicit need for consecutive independent trios. Any two consecutive independent trios can be decomposed into three independent pairs with no decrease in the number of demands covered, and a *possible* increase. Consider Figure 6.3.

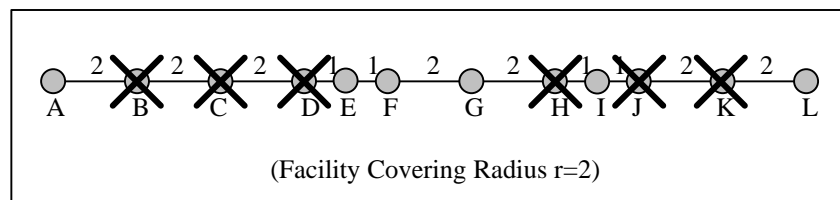


Figure 6.3 – Two Consecutive Independent Trios

In the worst-case, this can be decomposed into three independent pairs with no loss of demand coverage as shown in Figure 6.4.

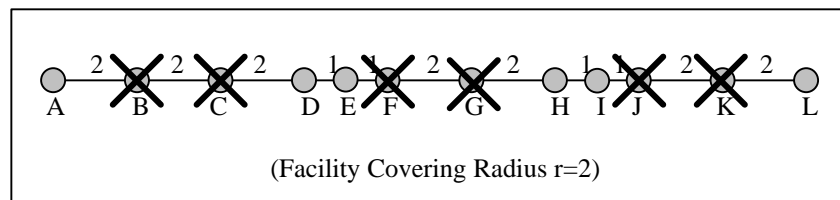


Figure 6.4 – Three Consecutive Independent Pairs

This decomposition will always provide at least equivalent coverage because the new facility Points F and G are (a) every demand in between themselves and the previous locations (D and H) and (b) each other. If condition (b) weren't true, the problem would have been decomposed into separate path MCP problems. Similar to the decomposition of independent groups of four or more facilities, it may result in increased coverage.

Note that while two consecutive independent trios can be decomposed into three consecutive independent pairs, the converse is not necessarily true. An example is shown in Figure 6.5.

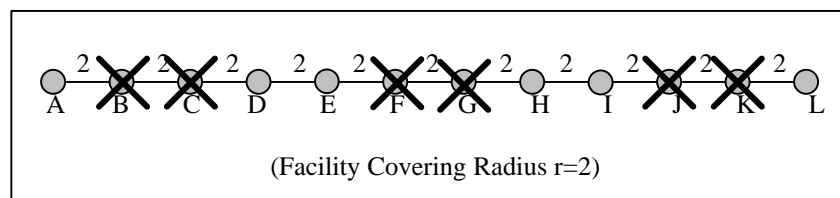


Figure 6.5 – Another Three Consecutive Independent Pairs

This optimal solution of $z=6$ facilities cannot be recomposed into two independent trios; such a combination could only cover the demands at Points A through J.

It is worthwhile to note that an optimal solution may have more than one independent trio. Unlike the case of uniform distances, the optimal solution to the MCP for paths with non-uniform link distances may have more than one independent trio. An example is shown in Figure 6.6.

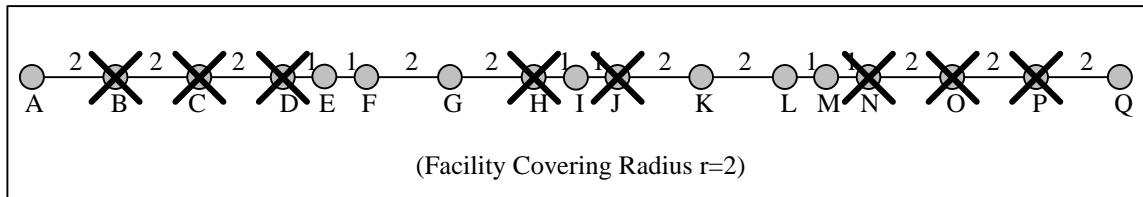


Figure 6.6 – More Than One Independent Trio in Optimal Solution

In this example, the optimal solution to the MCP has $z=8$ facilities comprised of two (2) independent trios and one (1) independent pair. These satisfy the equation

$$z^*=2a+3b \quad (6.2)$$

where $a=1$ and $b=2$. If it was attempted to decompose the solution into four (4) independent pairs, they could not cover all of the demands along the path. An attempt to increase the number of independent pairs will result in a solution that is no longer optimal, as shown in Figure 6.7.

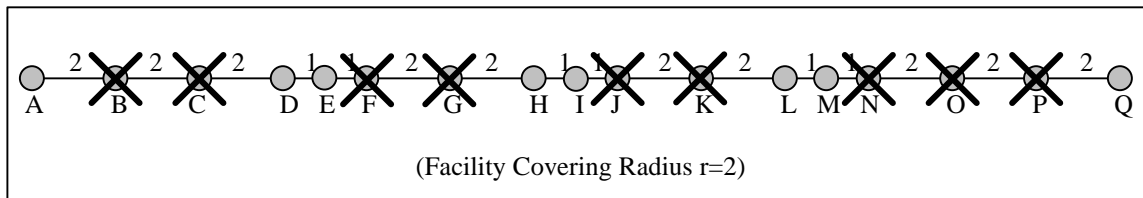


Figure 6.7 – Decomposition Attempt Resulting in Non-optimal Solution

For this instance, attempts to change the decomposition result in increasing the number of facilities required.

Note: If facilities were not restricted to vertex locations and could be located anywhere on the path, the optimal pattern would be similar to the uniform link distance case, in that the optimal solution would consist of independent pairs and at most one independent trio at the end of the path.

6.2 Simple Path Algorithm.

For non-uniform link distances, the dual objective of maximizing the length of the path covered by a fixed number of facilities is no longer analogous to maximizing the number of independent facility groups (Section 5.1). Due to this characteristic, there is not a closed form solution to determine the exact number of facilities required. In order to determine the optimal number and location of the facilities along a path with non-uniform link distances, a bounding enumeration technique may be executed as follows:

Step 1 (Initial Upper Bound): Start at one end of the path and place an independent pair of facilities to cover as many demands as possible. Repeat the process until either (1) the last demands on the path are covered with an independent pair, or (2) the last demands on the path are covered with an independent trio. The number of facilities required is z_{UB} , an initial upper bound. Store this solution as the incumbent.

Step 2 (Possible Improvement Combinations): Determine the number of (2,3) combinations for which $(z_{UB}-1)=2a+3b$ holds true. Eliminate combinations with consecutive independent trios. Table 6.1 shows eligible combinations for a sampling of z_{UB} values:

$(z_{UB}-1)$	Combinations (Line-through consecutive trios)	No. to Consider
2	(2)	1
3	(3)	1
4	(2-2)	1
5	(2-3), (3-2)	2
6	(2-2-2), (3-3)	1
7	(3,2,2), (2-3-2), (2-2-3)	3
8	(2-2-2-2), (2-3-3), (3,2,3), (3-3-2)	2
9	(2-2-2-3), (2-2-3-2), (2-3-2-2), (3-2-2-2), (3-3-3)	4
10	(2-2-2-2-2), (2-2-3-3), (2-3-2-3), (2-3-3-2), (3-3-2-2), (3-2-3-2), (3-2-2-3)	4
11	(2-2-2-2-3), (2-2-2-3-2), (2-2-3-2-2), (2-3-2-2-2), (3-2-2-2-2), (3-3-3-2), (3-3-2-3), (3-2-3-3), (2-3-3-3)	5
12	(2-2-2-2-2-2), (2-2-3-2-3), (2-3-2-2-3), (3-2-2-2-3), (2-3-2-3-2), (3-2-2-3-2), (3-2-3-2-2) (Consecutive trio combinations not shown here)	7
...

Table 6.1 – Combinations of z_{UB}

Step 3 (Enumeration and Improvement): For each permutation, locate the facilities accordingly in independent pairs and independent trios along the path (always starting with the same end of the path). If any of the combinations results in a successful MCP cover, the new upper bound is $(z_{UB}-1)$; store the new incumbent and go to Step 2. If none of the combinations result in a successful MCP cover, the incumbent is an optimal solution with $z^*=z_{UB}$.

As written, this algorithm works by examining incremental decreases in the objective function, then the combinations at each objective function value. Without further improvements, this algorithm fails to achieve polynomial complexity.

6.3 On the Optimality of All Independent Pairs.

If Step 1 of the Simple Path Algorithm is applied to a path with uniform link distances and it results in a solution of all independent pairs, that solution is guaranteed to be optimal (Section 5.1 and 5.2). In the case of non-uniform link distances, this characteristic does not hold true. After executing Step 1 of the Simple Path Algorithm, a solution of all independent pairs is only guaranteed to be an initial upper bound. The algorithm must be continued to examine decreased objective functions that will include independent trios.

Proof: By contradiction, the application of the Simple Path Algorithm to the path in Figure 6.8 (below) will yield an initial upper bound of $z_{UB}=16$ with eight (8) independent pairs. Since line symmetry exists, this result will be obtained no matter on which end the facilities are initially located.

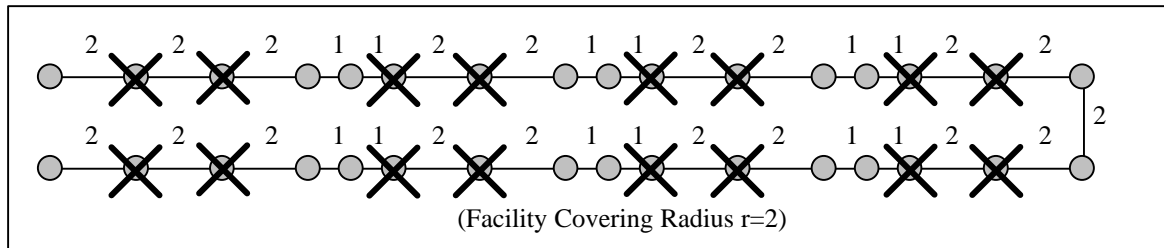


Figure 6.8 – Upper Bound with Independent Pairs

However, the optimal solution is $z^*=14$ as shown in Figure 6.9, consisting of four (4) independent pairs and two (2) independent trios.

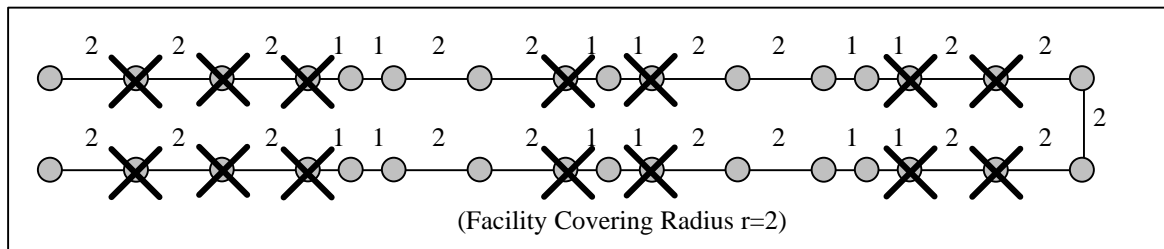


Figure 6.9 – Improvement to Solution over All Independent Pairs

6.4 Obtaining a Lower Bound for the Algorithm.

One method to improve the algorithm complexity is to develop a lower bound on the optimal solution based on the upper bound. The bound is as follows:

Theorem 6.2. Given an initial upper bound, z_{UB}^0 , to the optimal solution (z^*) obtained from Step 1 of the Simple Path Algorithm, the strict lower bound on the optimal solution (z^*) is:

$$z^* \geq z_{LB} = z_{UB}^0 - \lfloor (z_{UB}^0 - t)/6 \rfloor \quad (6.3)$$

where $t=3$ if an independent trio is placed during Step 1 of the Simple Path Algorithm; $t=0$ otherwise. Note: The term z_{UB}^0 is used to denote the initial upper bound because each incumbent solution in the Simple Path Algorithm is a new upper bound until the algorithm terminates.

Proof: The proof consists of four components:

- (1) Facility pattern for initial upper bound. When applying Step 1 from the Simple Path Algorithm, the initial upper bound, z_{UB}^0 , will be obtained by a pattern of facilities that will either be all independent pairs or one independent trio and all other groups as independent pairs. This pattern is defined in Section 6.2.
- (2) The minimum number of independent pairs with possible recomposition and improvement is three (3). Given the facility pattern for the initial upper bound, it is important to determine the minimum number of independent pairs that can be

recomposed to incorporate to (a) incorporate an independent trio, (b) decrease the number of facilities required, and (c) still cover all of the demands.

Obviously, an initial upper bound of $z_{UB}^0=2$ with one independent pair cannot be improved upon. Also, the solution with $z_{UB}^0=4$ obtained from Step 1 of the Simple Path Algorithm cannot be improved upon either; if $z^*=3$, the algorithm would have placed an independent trio in Step 1 instead of two pairs.

However, it is possible that a solution with $z_{UB}^0=6$ with three (3) independent pairs obtained from Step 1 of the Simple Path Algorithm can be improved upon. An example is shown in Figure 6.10. (The independent pairs were placed from left-to-right, requiring a total of three.)

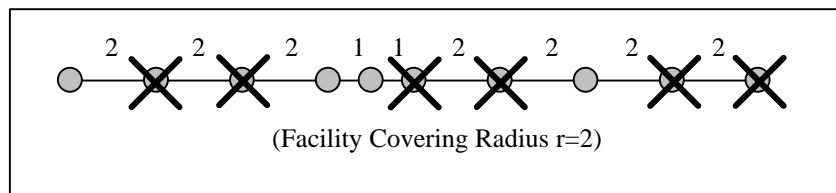


Figure 6.10 – Three Independent Pairs

Meanwhile, this solution can be recomposed to an improved $z^*=5$ with one independent pair and one independent trio as shown in Figure 6.11.

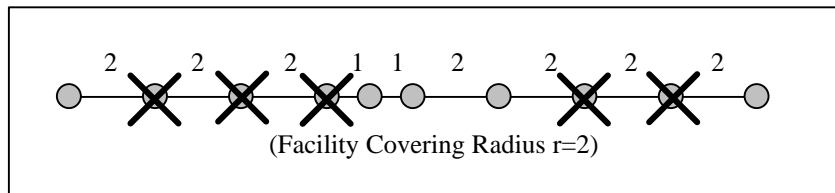


Figure 6.11 – Improved Recomposition

(3) Derivation of the Lower Bound. The maximum possible (though not necessarily likely) improvement from the initial upper bound, z_{UB}^0 , is to replace every three independent pairs (6 facilities) with an independent pair and an independent trio (5 facilities). Therefore, the initial upper bound, z_{UB}^0 , can be decreased by at most $n = \lfloor (z_{UB}^0 - t)/6 \rfloor$ facilities. Therefore, $z^* \geq z_{LB} = z_{UB}^0 - \lfloor (z_{UB}^0 - t)/6 \rfloor$.

(4) The requirement for no consecutive independent trios does not further restrict the lower bound. The initial upper bound has a solution with at most one independent trio, and the lower bound is obtained by replacing three pairs with an independent pair and an independent trio. Therefore, if the lower bound is feasible it will have at most one more independent trio than independent pairs of facilities; it is possible for these groups to be ordered such that no independent trios are adjacent.

The convenient result of this bound is that the application of the Step 1 from the Simple Path Algorithm without seeking optimality with the remaining steps will result in a solution that is no worse than 20% larger than the optimal solution.

6.5 Revised Path Algorithm.

Incorporating the lower bounds, the algorithm can be revised as follows:

Step 1 (Initial Upper Bound): Start at one end of the path and place an independent pair of facilities to cover as many demands as possible. Repeat the process until either (1) the last demands on the path are covered with an independent pair, or (2) the last demands on the path are covered with an independent trio. The number of facilities required is z_{UB}^0 , an initial upper bound. Store this solution as the incumbent. Calculate and store the lower bound of $z_{LB} = z_{UB}^0 - \lfloor (z_{UB}^0 - t)/6 \rfloor$.

Step 2 (Termination on Lower Bound): Given z_{UB} , the current incumbent.

If $z_{UB} = z_{LB}$, then $z^* = z_{UB}$. The incumbent is the optimal solution.

Step 3 (Determine Combinations for Possible Improvement): Determine the number of (2,3) combinations for which $(z_{UB}-1)=2a+3b$ holds true. Eliminate combinations with consecutive independent trios. (Reference Table 6.1.)

Step 4 (Enumeration and Improvement): For each permutation, locate the facilities accordingly in independent pairs and independent trios along the path (always starting with the same end of the path). If any of the combinations results in a successful

MCP cover, the new upper bound is $(z_{UB}-1)$; store the new incumbent and go to Step

2. If none of the combinations result in a successful MCP cover, the incumbent is an optimal solution with $z^*=z_{UB}$.

6.6 Revised Path Algorithm Complexity.

The general form for the algorithm complexity is:

$$O(\bullet) = f[1+\sum g(z)] \quad \text{where } z = (z_{UB}^0-1), (z_{UB}^0-2), \dots, (z_{LB}) \quad (6.4)$$

where f represents the complexity of locating a given combination of facilities along a path to determine its feasibility, the value 1 represents the initial placement of facilities to determine z_{UB} , and $g(z)$ represents the number of combinations for a given number of facilities (z) be examined. In order to examine the overall complexity, these components must be examined in further detail.

6.6.1 Complexity of locating a given combination of facilities (l).

For a given combination of facilities, the number of possible facility locations (N) must each be examined to determine whether each requires a facility to be placed, resulting in exactly N binary decisions for each application of a combination of pairs and trios. With $l(\bullet)=N^*(\bullet)$, the complexity can be rewritten in a more simplified form as:

$$O(\bullet) = N[1 + \sum g(z)] \quad \text{where } z = (z_{UB}^0 - 1), (z_{UB}^0 - 2), \dots, (z_{LB}) \quad (6.4)$$

6.6.2 The Number of Combinations for a Given Number of Facilities, $g(z)$.

For a given z , there exist two cases of significance to determine the number of combinations for which it can be decomposed into ($z=2a+3b$). When the value of z is even, the independent groups can be decomposed into pairs and trios as follows:

$$b = 0, 2, 4, \dots, \lfloor z/3 \rfloor \quad (6.5)$$

$$a = (z-3b)/2 \quad (6.6)$$

When the value of z is odd, the pair/trio decompositions are:

$$b = 1, 3, 5, \dots, \lfloor z/3 \rfloor \quad (6.7)$$

$$a = (z-3b)/2 \quad (6.8)$$

For a given decomposition of (z) into values (a) and (b), the number of combinations of their placement along the path without consecutive independent trios is bounded by the number of combinations of their placement along the path allowing for consecutive independent trios. This bound is determined as:

$$g(z:a,b) = [(a+b)!/(a!b!)] \quad (6.9)$$

Although the value $g(z;a,b)$ is a function of $N!$, it is not consistently maximized for a given proportion of a and b , so the general form will remain as:

$$g(z) = \sum_{(a,b)} [(a+b)!/(a!b!)] \quad (6.10)$$

6.6.3 The Value of z which Maximizes $g(z)$.

Since the upper bound on $g(z)$ is relaxed to allow for consecutive trio combinations, $g(z)$ strictly increases with the value for z . Therefore the relaxation inherent in the calculation of $g(z)$ can simplify the closed form for the complexity as:

$$O(\bullet) \leq N \{ 1 + [(z_{UB}^0 - z_{UB}^0 + 1) * g(z_{UB}^0 - 1)] \} \quad (6.11)$$

6.6.4 Complexity Bound after Applying Step 1 of the Revised Path Algorithm.

Summarizing the results of previous sections, the complexity of the Revised Path Algorithm is bounded by

$$O(\bullet) \leq N \{ 1 + \lfloor (z_{UB}^0 - t)/6 \rfloor * \sum_{(a,b)} [(a+b)!/(a!b!)] \} \quad (6.12)$$

where

$N =$ # of facility locations / demands

$z_{UB}^0 =$ Upper bound on # of facilities determined from Step 1

$$\begin{aligned}
t &= \begin{array}{l} 3 \text{ if an independent trio is placed in Step 1} \\ 0 \text{ otherwise} \end{array} \\
b &= \begin{array}{l} 1, 3, 5, \dots, \lfloor (z_{UB}^0 - 1)/3 \rfloor \text{ if } (z_{UB}^0 - 1) \text{ is odd} \\ 0, 2, 4, \dots, \lfloor (z_{UB}^0 - 1)/3 \rfloor \text{ otherwise} \end{array} \\
a &= (z_{UB}^0 - 1 - 3b)/2
\end{aligned}$$

6.6.5 Overall Complexity Bound for the Revised Path Algorithm.

Efforts at closed form analysis produced a series of equations that differed based on the initial value of N and only simplified the complexity to “ $O_{RPA}(\bullet) = N^2 \cdot (\text{a combinatorial quantity})$.” Efforts to reduce the “combinatorial quantity” to a polynomial expression were unsuccessful.

On the other hand, efforts to prove the Modified Covering Problem as NP-complete via mathematical proof were also unsuccessful.

However, an alternative approach did provide additional insight into complexity of the Revised Path Algorithm. The worst-case complexity bound from Equation (6.12) was applied for values of $N=2$ through $N=120$. Then a series of curve-fitting applications were applied to approximate the worst-case complexity as a function of the number of demands, N .

Initially, only the following simple functions and their fits are shown:

Polynomial: # of Operations = $N^{(4.7083411655)}$ (6.13)

Exponential: # of Operations = 1.20664390102^N (6.14)

The results of the curve fits are shown in Figures 6.12 and 6.13 below:

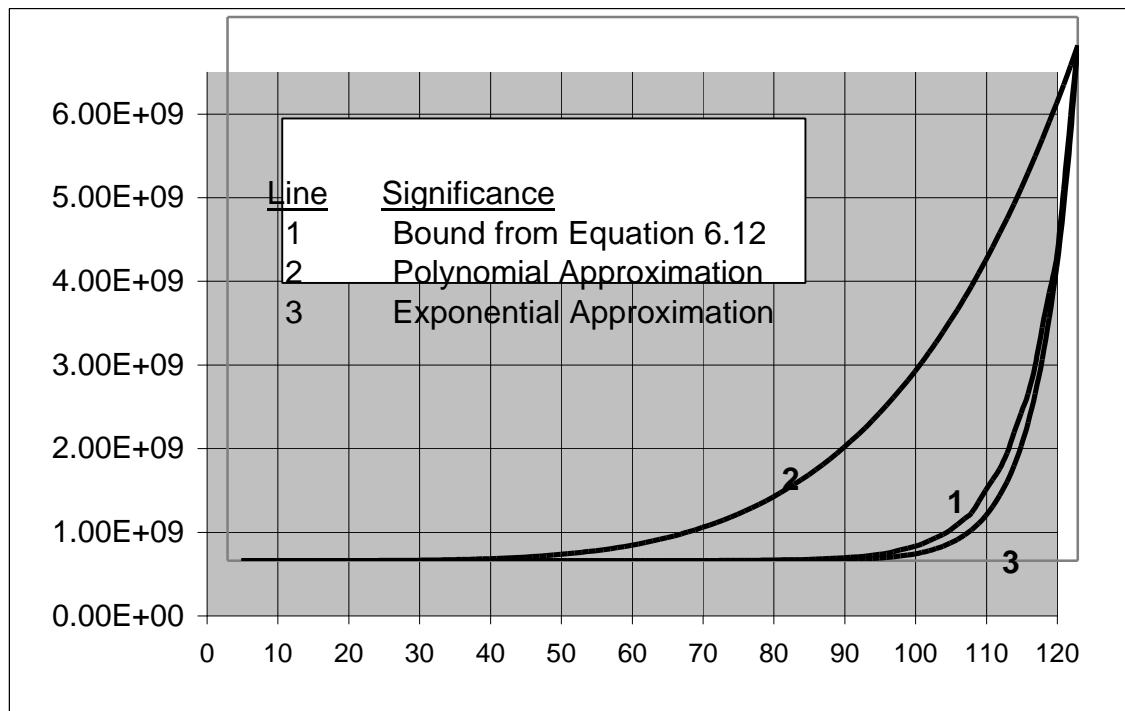


Figure 6.12 – Plot of Complexity Curve and Approximations

Although the exponential approximation appears to better fit the worst-case complexity of the Revised Path Algorithm, it merits analysis on a logarithmic scale.

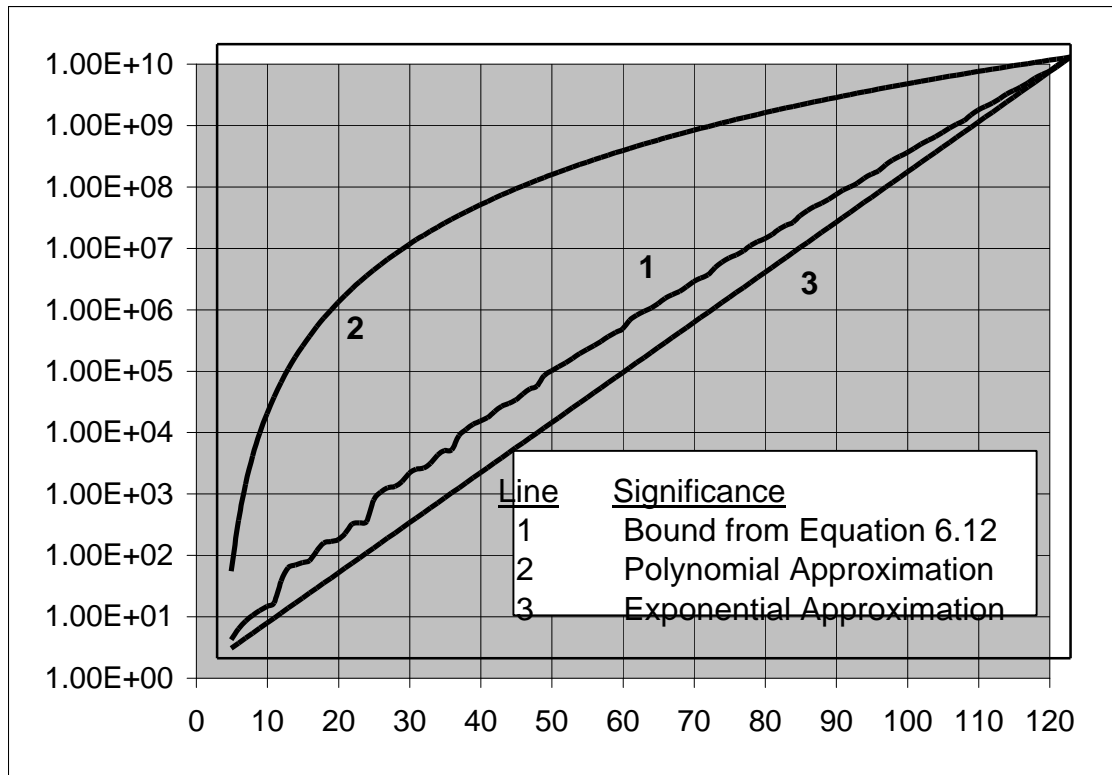


Figure 6.13 – Logarithmic Plot of Complexity Curve and Approximations

From Figure 6.13, the complexity of the Revised Path Algorithm appears to be display non-polynomial behavior.

Next, a series of efforts were applied to determine fit the curve for the Revised Path Algorithm with the constraint that the curve equation be of the form:

$$O_{RPA}(\bullet) = N^{2*}(\text{an expression}) \quad (6.15)$$

The most successful fit occurred for the form:

$$O_{RPA}(\bullet) = N^{2*}(A*B^{(CN)}) \quad (6.16)$$

with the following solution:

$$O_{RPA}(\bullet) \approx 0.02161373 * N^2 * (1.149166593)^{(1.006434497N)} \quad (6.17)$$

With the following successful curve fits as shown in Figures 6.14 and 6.15:

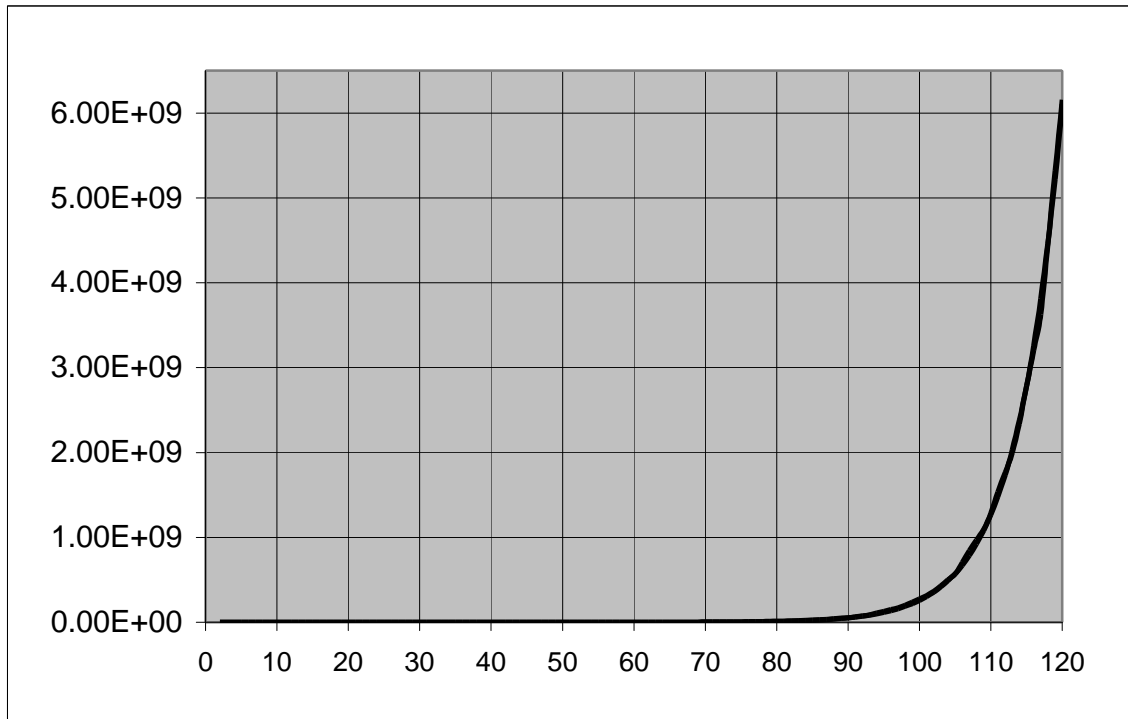


Figure 6.14 – Plot of Successful Curve Fit

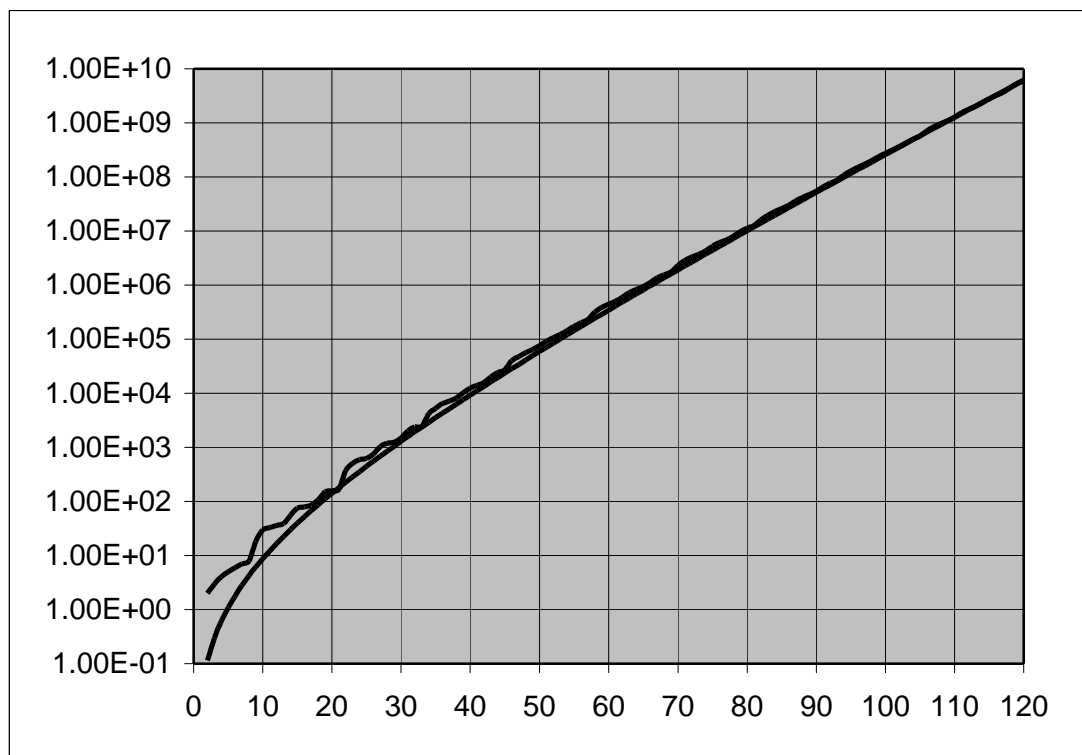


Figure 6.15 – Logarithmic Plot of Successful Curve Fit

CHAPTER 7

THE MODIFIED COVERING PROBLEM APPLIED TO TREES

This chapter develops a bound for the optimal solution of the modified covering path on trees based on (1) the optimal solution to a decomposition of the tree into paths and (2) the degree of the nodes within the tree.

7.1 Preliminary Definitions.

(1) The term *tree* refers to a connected graph that contains no cycle [Ahuja, 28].

It has the properties common to all trees, namely:

- a) A tree on n nodes contains exactly $n-1$ arcs.
- b) A tree has at least two leaf nodes (i.e., nodes with degree 1)
- c) Every two nodes of a tree are connected by a unique path.

Additionally, no link distance between two connected nodes is larger than the covering radius of the facility being examined for the Modified Covering Problem. If this last component is violated, the MCP is accordingly decomposed into two separable sub-problems for each such link.

(2) An *interior node* on a path or tree is a node with degree of $d \neq 1$.

(3) An *exterior node* on a path or tree is a node with degree of $d=1$.

(4) *Tree decomposition* refers to performing an arc partition of the tree in order to form the minimum numbers of paths. The intersection of these paths may include common node(s), but will contain no common arcs. The union of these paths is the original tree prior to decomposition.

7.2 Tree Decomposition and Motivation for Bounding.

Previous effort has shown that an optimal solution can be reached on paths, though not necessarily in polynomial time. A logical approach to solving the MCP on trees is to (1) decompose the tree into paths, (2) find the optimal solution for these paths, and (3) adjust the ordering of independent pairs and trios along the paths to take advantage of the paths' intersections and reduced the size of the solution by eliminating redundant facilities.

The difficulty with this approach is its complexity. The cardinality of the number of paths into which a tree can be decomposed (arc-partitioned) is:

$$N_P = |N_{d=ODD}|/2 = (\# \text{ of nodes of odd degree})/2 \quad (7.1)$$

However, as trees increase in size, the number of independent, unique decompositions increases in a combinatorial manner. Second, if a path intersects with more than one

other path in the tree, its independent groups of facilities can be rearranged to allow for different combinations of facility redundancy. Proceeding with this solution approach appears more difficult than strict enumeration... and may not offer any benefit in complexity.

However, tree decomposition is very useful in bounding the optimal solution. An arbitrary tree decomposition and solution of independent paths can provide an upper bound for the optimal solution. With further inspection of the effects of intersecting paths, a lower bound can be obtained as well.

7.3 Intersecting Paths at Interior Nodes.

Theorem 7.1. Within the MCP framework, consider two paths with optimal solutions, z_1^* and z_2^* . The intersection of two paths at interior nodes will result in a tree with an optimal solution that is between zero and two facilities less than the sum of the independent path optimal solutions.

Proof: The proof consists of showing the three possible effects of intersecting paths at interior nodes, as well as the limits to improvement:

- (1) No Improvement. The following example in Figure 7.1 shows the optimal MCP solution to two paths that, when intersected at interior nodes, results in no

improvement to the number of facilities required for the tree. Path #1 and Path #2 represent a tree that has been decomposed; the common node (Node E) represents the intersection of the paths within the tree. When solved separately, Path #1 and Path #2 each require four (4) facilities. When these solutions are conjoined to re-form the tree, there is no facility redundancy. The tree requires eight (8) facilities, the sum of the two path solutions.

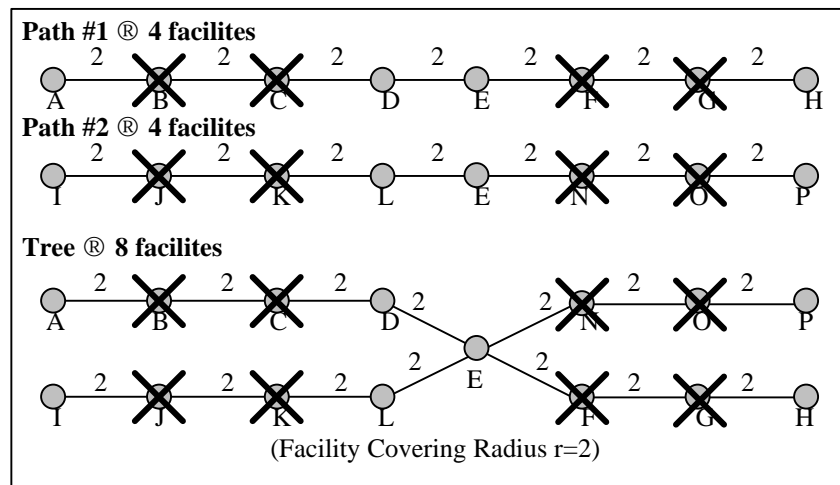


Figure 7.1 – No Improvement with Intersecting Paths

- (2) One (1) Facility Improvement. Similarly, Figure 7.2 shows an improvement of one less facility required to solve the MCP for the tree. Path #1 and Path #2 again represent a tree that has been decomposed, with the common Node C. Path #1 and #2 each require two (2) facilities when solved separately. When they are conjoined to re-form the tree, however, there is one (1) redundant facility. The tree requires only three (3) facilities.

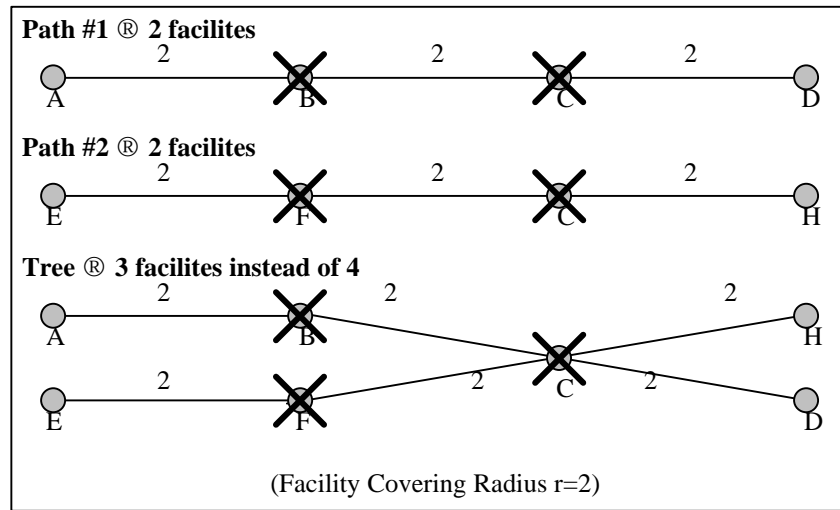


Figure 7.2 – One Facility Improvement with Intersecting Paths

- (3) Two (2) Facility Improvement. Figure 7.3 shows an improvement of two fewer facilities. Path #1 and #2 represent a decomposed tree, with the common Node B. Path #1 and #2 each require two (2) facilities when solved separately. When they are conjoined to re-form the tree, however, there are two (2) redundant facilities: B and either A or D. The tree requires only two (2) facilities.

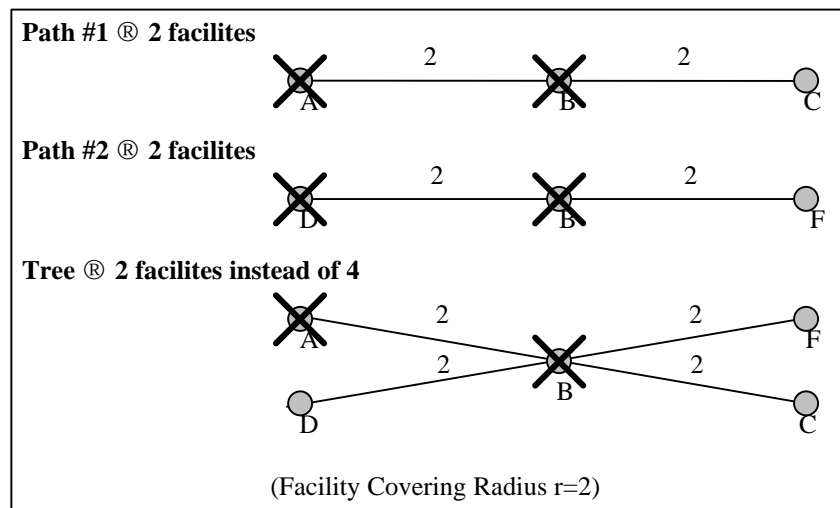


Figure 7.3 – Two Facility Improvement with Intersecting Paths

(4) No facility improvements of more than two are possible. If a tree is decomposed into two paths, the optimal path solutions will contain at most two redundant facilities when re-formed as the tree. Take all of the facilities from Path #1 to remain in the tree solution, thereby covering all nodes that were on Path #1.

(a) Case #1 – The Path #2 solution also requires a facility at the common node. If the Path #1 and Path #2 solutions both required a facility on their common node in their separate solutions, one of those facilities is redundant and may be removed from the conjoined (tree) solution, for a total of one (1) redundant facility.

If the Path #2 solution required another facility only for the purpose of covering its facility at the common node in the path solution, it may also be removed, for a total of two (2) redundant facilities.

However, the Path #2 solution would not require more than one facility for the sole purpose of covering its facility at the common node. By definition, the existence of such a facility would be redundant in the path solution and the Path #2 solution would not be optimal. Therefore, any other facilities on the Path #2 solution are required to cover demands that are beyond the covering radius distance from the common node. None of those facilities may be eliminated in the tree solution.

- (b) Case #2 – The Path #2 solution does not require a facility at the common node. If the common node is in between two independent groups in Path #2, two sub-scenarios exist. First, if the Path #1 facility at the common node in the tree is in between two independent pairs from the Path #2 solution, no redundant facilities exist. Elimination of even one facility will result in an uncovered facility. Second, if the common node is in between an independent pair and an independent trio, at most the closest facility in the independent trio may become redundant – if the nodes it was required to cover are now covered by a the Path #1 facility at the common node. Of note, it was shown earlier that an optimal path solution will not contain (1) two consecutive trios or (2) independent groups of size larger than three... so no other cases exist.

7.4 Intersecting Paths at Interior/Exterior Nodes.

Theorem 7.2. Within the MCP framework, consider two paths with optimal solutions, z_1^* and z_2^* . The intersection of two paths at one interior node and one exterior node will result in a tree with an optimal solution that is between zero and two facilities less than the sum of the independent path optimal solutions.

Proof: This proof also consists of showing the three possible effects of intersecting paths at an interior and exterior nodes, as well as limits to solution improvement:

- (1) No Improvement. The example in Figure 7.4 shows the optimal MCP solution to two paths that, when intersected at an interior node and an exterior node, results in no improvement to the number of facilities required for the tree. Path #1 and Path #2 represent a decomposed tree. They have a common Node D, the intersection of the paths within the tree. When solved separately, Path #1 and Path #2 each require four (4) and two (2) facilities, respectively. When these solutions are conjoined to re-form the tree, there is no facility redundancy. The tree requires six (6) facilities, the sum of the two path solutions.

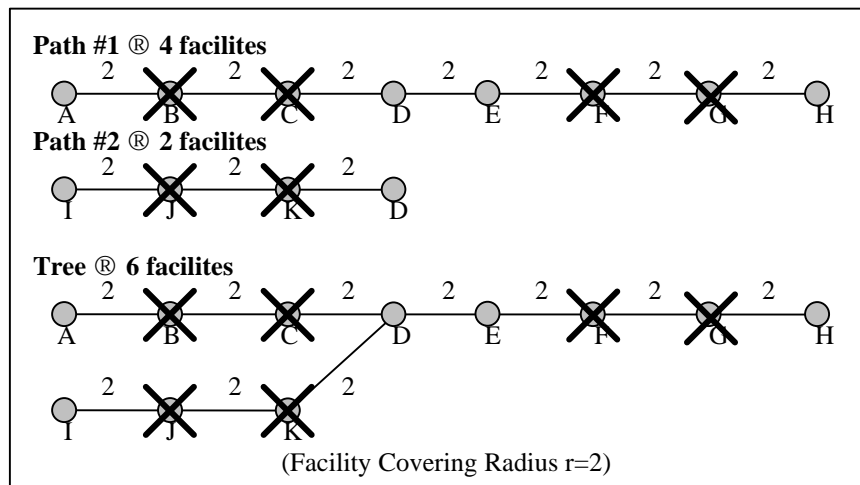


Figure 7.4 – No Improvement with Intersecting Paths

- (2) One (1) Facility Improvement. Figure 7.5 shows the optimal MCP solution to two paths that result in one redundant facility for the optimal MCP solution for the tree. Path #1 and Path #2 represent a decomposed tree with their common Node C. When solved separately, Path #1 and Path #2 each require two (2)

facilities. When these solutions are conjoined, there is one redundant facility. The tree requires three (3) facilities, one less than the sum of the two path solutions.

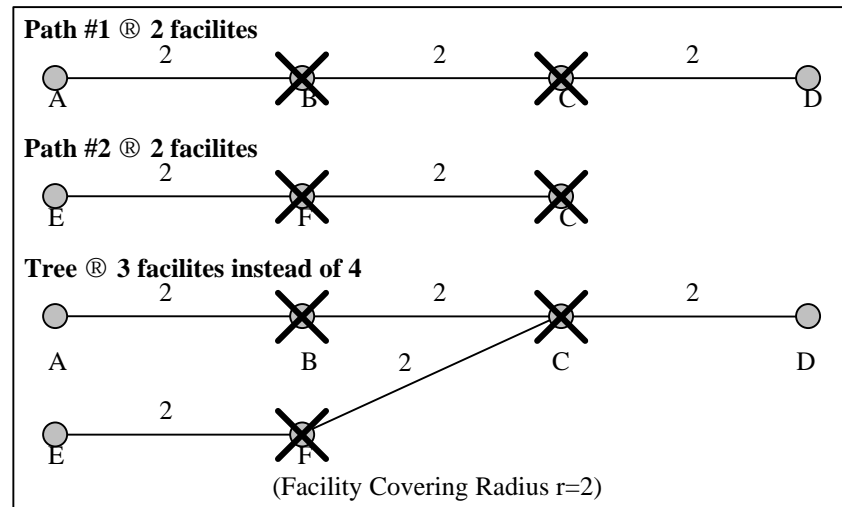


Figure 7.5 – One Facility Improvement with Intersecting Paths

- (3) Two (2) Facility Improvement. This example shows an improvement of two fewer facilities required to solve the MCP for the tree. Figure 7.6 shows an improvement of two fewer facilities. Path #1 and #2 represent a decomposed tree, with the common Node B. Path #1 and #2 each require two (2) facilities when solved separately. When they are conjoined, there are two (2) redundant facilities: B and either A or D. The tree requires only two (2) facilities.

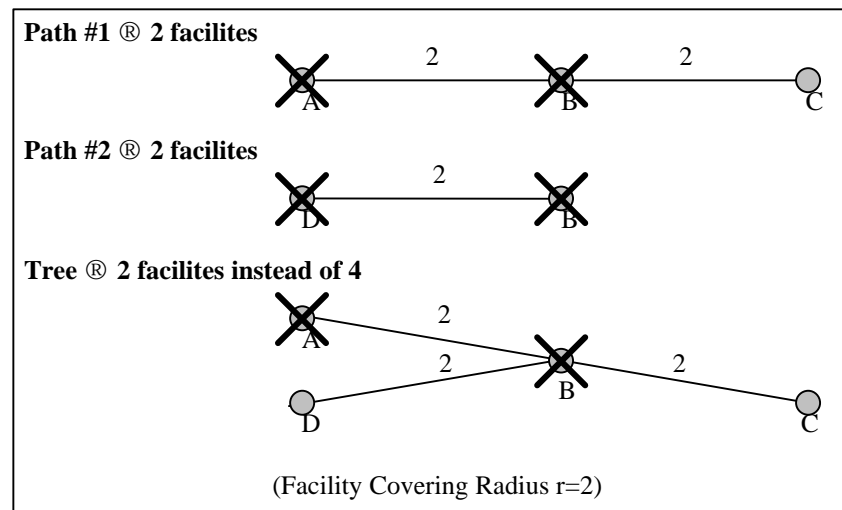


Figure 7.6 – Two Facility Improvement with Intersecting Paths

- (4) No facility improvements of more than two are possible. Assign Path #1 as the path with the common node as one of its interior nodes. A facility at the common node in the Path #1 solution would provide the most opportunity for redundancy of Path #2 facilities in the conjoined solution. If this case can be shown to allow no more than two redundant facilities in the Path #2 solution, it is the only case than must be examined. A Path #1 solution without a facility at the common node can have no greater effect on the redundancy of facilities in a Path #2 solution.

The existence of a facility at an interior node in the Path #1 solution indicates that it is required to cover demands, not just another facility. Accordingly, all facilities from the Path #1 solution may be taken as part of the optimal tree solution.

- (a) Case #1 – The Path #2 solution also requires a facility at the common node.

Similar to the interior point case, this facility from the Path #2 solution is redundant. If the Path #2 solution has another facility that exists only to cover the facility at the common node, that facility will also be redundant in the tree solution, for a total improvement of two (2) facilities over the sum of the path solutions. No other facilities in the Path #2 solution may be redundant, as their existence would only be required to cover demands as well as other facilities.

- (b) Case #2 – The Path #2 solution does not require a facility at the common

node. In this case, the Path #1 facility at the common node can make at most the closest facility in an independent trio redundant. Removal of more than one facility of an independent trio or one facility of an independent pair from the Path #2 solution that covers the common node would result in an uncovered facility, if not uncovered demands as well.

Note: The intersection of two paths at exterior nodes results in a path, not a tree. It is not a situation that results from decomposing a tree into the minimum number of independent paths (N_p), and therefore does not require examination here.

7.5 Determining the Bounding Effects of Each Node.

Theorems 7.1 and 7.2 can be extended to determine the bounding effects of intersection nodes within the tree for comparing the optimal MCP solution to the tree versus the optimal MCP solution to independent paths. As more than two paths intersect a node, its degree increases and the possible redundancy of facilities when the paths are conjoined also increases. The maximum facility redundancy for independent paths based on the node degree is shown in Table 7.1.

Node Degree	1	2	3	4	5	6	7	8	9	10	11	12	13	...
Min. No. of Paths through Node	1	1	2	2	3	3	4	4	5	5	6	6	7	...
Maximum Facility Redundancy	0	0	2	2	4	4	6	6	8	8	10	10	12	...

Table 7.1 – Maximum Facility Redundancy at a Node where Paths Intersect

Accordingly, the maximum improvement to an optimal MCP tree solution over the optimal MCP solution for its path decomposition due to a single node is:

$$\text{MAX } \Delta_{Z_N} = 2 * (\lceil (d_N/2) \rceil - 1) \quad (7.2)$$

where d_N represents the degree of the node.

7.6 Closed Form Bound for the MCP for Trees.

The closed form bound for the optimal solution to the MCP for a tree is:

$$\Sigma_P(z_P)^* - \Sigma_N[2^* \lceil (d_N/2) \rceil - 2] \leq z_T^* \leq \Sigma(z_P)^* \quad (7.3)$$

where

$z_T^* \equiv$	Optimal MCP solution for the tree
$\Sigma_P(z_P)^* \equiv$	Sum of facilities required to solve the MCP problem for the tree when decomposed into the minimum number of paths, N_P .
$\Sigma_N[2^* \lceil (d_N/2) \rceil - 2] \equiv$	Sum of maximum facilities redundancies over all nodes when conjoining the independent path MCP solutions

This result is most useful for evaluating if there *may* exist sufficient benefit in reducing the number of facilities to justify additional time and effort to seek improvements to the decomposed path solution.

CHAPTER 8

CONCLUSIONS AND FUTURE RESEARCH

8.1 Conclusions.

The Modified Covering Problem is both relevant and important to our society within the military and civil defense applications. Solving -- instead of satisficing -- the problem provide sufficient financial incentive for the problem's current and continued study.

Within the framework of the Modified Covering Problem, optimality of all independent pairs and at most one independent trio of facilities is guaranteed for a path with uniform link distances. This solution can be obtained in polynomial time of $O(n)$.

The same polynomial algorithm can solve within 20% of optimality the MCP on paths with non-uniform link distances. Meanwhile, the Revised Path Algorithm is guaranteed to obtain the optimal MCP solution for a path with non-uniform distances. However, the algorithm's worst-case complexity as shown in equation (6.18) is exponential rather than polynomial.

Finally, the Revised Path Algorithm can be combined with (1) tree decomposition via arc partitioning and (2) knowledge of the nodes' degrees within a tree to provide strong upper and lower bounds for the optimal MCP solution for a general tree.

8.2 Future Research.

Three areas merit additional research and development. First, the MCP should be examined for facilities with capacities to closer reflect reality. For example, artillery units usually have a maximum number of preplanned targets registered in their fire control computers. They can fire at any target within their range fan, allowing for high-altitude fire to overcome geographical barriers. However, they can only *rapidly* respond to requests for fire to targets at or near the preplanned targets. Obviously, the capacity for rapid response is more important than the artillery units theoretical infinite capacity when life and limb is at stake.

Second, general graphs should be examined. These would more accurately reflect the inter-site distances and possibly reduce the cardinality of the corresponding tree solution.

Finally, other algorithmic approaches should be examined in order to improve their complexity and provide more insight for solving the general network problems.

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THE MODIFIED COVERING PROBLEM

ON PATHS AND TREES

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The University of Arizona, 2001

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The Modified Covering Problem (MCP) is introduced and theory is developed for solving it on paths and trees. First, the Modified Covering Problem is defined as a subset of the Conditional Covering Problem, and motivations are proposed for its study. Next, a literature review examines relevant, published material.

The MCP is then formulated as a binary integer program, followed by an examination of the characteristics of its feasible solutions, optimality, and overall complexity. A polynomial algorithm is developed for the solving the MCP on paths with uniform link distances, and solving within 20% of optimality on paths with non-uniform link distances. Next, an exponential algorithm is developed to solve non-uniform link distance problems to optimality. The theory is then further expanded to construct an algorithm to develop strong upper and lower bounds for the optimal solution on trees with non-uniform link distances.